THE FIRST INITIAL BOUNDARY VALUE PROBLEM FOR HESSIAN EQUATIONS OF PARABOLIC TYPE ON RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we are concerned with the first initial boundary value problem for a class of fully nonlinear parabolic equations on Riemannian manifolds. As usual, the establishment of the *a priori* C^2 estimates is our main part. Based on these estimates, the existence of classical solutions is proved under conditions which are nearly optimal.

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1. Introduction

In this paper, we study the Hessian equations of parabolic type of the form

$$f(\lambda(\nabla^2 u + \chi), -u_t) = \psi(x,t) \tag{1.1}$$

in $M_T = M \times (0,T] \subset M \times \mathbb{R}$ satisfying the boundary condition

$$u = \varphi \text{ on } \mathcal{P}M_T, \tag{1.2}$$

where (M,g) is a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M and $\bar{M} := M \cup \partial M$, $\mathcal{P}M_T = BM_T \cup SM_T$ is the parabolic boundary of M_T with $BM_T = M \times \{0\}$ and $SM_T = \partial M \times [0,T]$, f is a symmetric smooth function of $n+1$ variables defined in an open convex symmetric cone $\Gamma \subset \mathbb{R}^{n+1}$ with vertex at the origin and

$$\Gamma_{n+1} \equiv \{\lambda \in \mathbb{R}^{n+1} : \text{ each component } \lambda_i > 0,\ 1 \leq i \leq n+1\} \subseteq \Gamma,$$

$\nabla^2 u$ denotes the Hessian of $u(x,t)$ with respect to $x \in M$, $u_t = \frac{\partial u}{\partial t}$ is the derivative of $u(x,t)$ with respect to $t \in [0,T]$, χ is a smooth (0, 2) tensor on $\bar{M}$ and $\lambda(\nabla^2 u + \chi) = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$ denotes the eigenvalues of $\nabla^2 u + \chi$ with respect to the metric g.

As in [4] (see [9] also), we assume that f satisfies the following structural conditions:

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n+1, \tag{1.3}$$

$$f \text{ is concave in } \Gamma \tag{1.4}$$

and

$$\delta_{\psi,f} \equiv \inf_M \psi - \sup_{\partial\Gamma} f > 0, \text{ where } \sup_{\partial\Gamma} f \equiv \sup_{\lambda_0 \in \partial\Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda). \tag{1.5}$$

We mean an admissible function by $u \in C^2(M_T)$ satisfying $(\lambda(\nabla^2 u + \chi), -u_t) \in \Gamma$ in M_T, where $C^k(M_T)$ denotes the space of functions defined on M_T which are

k-times continuously differentiable with respect to $x \in M$ and $[k/2]$-times continuously differentiable with respect to $t \in (0, T]$ and $[k/2]$ is the largest integer not greater than $k/2$. We note that (1.1) is parabolic for admissible solutions (see [4]).

We first recall the following notations

$$|u|_{C^k(\overline{M_T})} = \sum_{|\beta|+2r \leq k} \sup_{\overline{M_T}} |\nabla^\beta D_t^r u|,$$

$$\begin{aligned} |u|_{C^{k+\alpha}(\overline{M_T})} = {} & |u|_{C^k(\overline{M_T})} \\ & + \sup_{|\beta|+2r=k} \sup_{\substack{(x,s),(y,t) \in \overline{M_T} \\ (x,s) \neq (y,t)}} \frac{|\nabla^\beta D_t^r u(x,s) - \nabla^\beta D_t^r u(y,t)|}{\left(|x-y| + |s-t|^{1/2}\right)^\alpha} \end{aligned}$$

and $C^{k+\alpha}(\overline{M_T})$ denotes the subspace of $C^k(\overline{M_T})$ defined by

$$C^{k+\alpha}(\overline{M_T}) := \{u \in C^k(\overline{M_T}) : |u|_{C^{k+\alpha}(\overline{M_T})} < \infty\}.$$

In the current paper, we are interested in the existence of admissible solutions to (1.1)-(1.2). The key step is to establish the *a priori* C^2 estimates. Using the methods from [10], where Guan studied the elliptic counterpart of (1.1):

$$f(\lambda(\nabla^2 u + \chi)) = \psi(x) \tag{1.6}$$

in M satisfying the Dirichlet boundary condition, we are able to obtain these estimates under nearly minimal restrictions on f.

Our main results are stated in the following theorem.

Theorem 1.1. *Suppose that* $\psi \in C^\infty(\overline{M_T})$, $\varphi \in C^\infty(\overline{\mathcal{P}M_T})$ *for* $0 < T \leq \infty$,

$$(\lambda(\nabla^2\varphi(x,0) + \chi(x)), -\varphi_t(x,0)) \in \Gamma \text{ for all } x \in \overline{M} \tag{1.7}$$

and

$$f(\lambda(\nabla^2\varphi(x,0) + \chi(x)), -\varphi_t(x,0)) = \psi(x,0) \text{ for all } x \in \partial M. \tag{1.8}$$

In addition to (1.3)-(1.5)*, assume that*

$$f_j(\lambda) \geq \nu_0\Big(1 + \sum_{i=1}^{n+1} f_i(\lambda)\Big) \text{ for any } \lambda \in \Gamma \text{ with } \lambda_j < 0, \tag{1.9}$$

for some positive constant ν_0,

$$\sum_{i=1}^{n+1} f_i\lambda_i \geq -K_0 \sum_{i=1}^{n+1} f_i, \quad \forall \lambda \in \Gamma \tag{1.10}$$

for some $K_0 \geq 0$ *and that there exists an admissible subsolution* $\underline{u} \in C^2(\overline{M_T})$ *satisfying*

$$\begin{cases} f(\lambda(\nabla^2\underline{u} + \chi), -\underline{u}_t) \geq \psi(x,t) & \text{in } M_T, \\ \underline{u} = \varphi & \text{on } SM_T, \\ \underline{u} \leq \varphi & \text{on } BM_T. \end{cases} \tag{1.11}$$

Then there exists a unique admissible solution $u \in C^\infty(\overline{M_T})$ *of* (1.1)-(1.2).

Remark 1.2. Condition (1.9) is only used to derive the gradient estimates as many authors, see [21], [14], [8], [20], [23] and [27] for examples.

Condition (1.10) is used in the estimates for both $|\nabla u|$ and $|u_t|$. We will see that in the gradient estimates, condition (1.10) can be weakened by

$$\sum_{i=1}^{n+1} f_i \lambda_i \geq -K_0 \Big(1 + \sum_{i=1}^{n+1} f_i\Big), \quad \forall \lambda \in \Gamma. \tag{1.12}$$

As in [10], the existence of $\underline{u}$ is useful to construct some barrier functions which are crucial to our estimates.

The most typical examples of f satisfying the conditions in Theorem 1.1 are $f = \sigma_k^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \leq l < k \leq n+1$, defined in the Gårding cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^{n+1} : \sigma_j(\lambda) > 0, j = 1, \ldots, k\},$$

where σ_k are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}, \quad k = 1, \ldots, n+1.$$

When $f = \sigma_{n+1}^{1/(n+1)}$, equation (1.1) can be written as the parabolic Monge-Ampère equation:

$$- u_t \det(\nabla^2 u + \chi) = \psi^{n+1}, \tag{1.13}$$

which was introduced by Krylov in [17] when $\chi = 0$ in Euclidean space. Our motivation to study (1.1) is from their natural connection to the deformation of surfaces by some curvature functions. For example, equation (1.13) plays a key role in the study of contraction of surfaces by Gauss-Kronecker curvature (see Firey [6] and Tso [25]). For the study of more general curvature flows, the reader is referred to [1], [2], [15], [22] and their references. (1.13) is also relevant to a maximum principle for parabolic equations (see Tso [26]).

In [21], Lieberman studied the first initial-boundary value problem of equation (1.1) when $\chi \equiv 0$ and ψ may depend on u and ∇u in a bounded domain $\Omega \subset \mathbb{R}^{n+1}$ under various conditions. Jiao and Sui [16] considered the parabolic Hessian equation of the form

$$f(\lambda(\nabla^2 u + \chi)) - u_t = \psi(x, t) \tag{1.14}$$

on Riemannian manifolds using techniques from [10] and [11] where the authors studied the corresponding elliptic equations. Guan, Shi and Sui [13] extended the work of [16] using the idea of [10]; they also treated the parabolic equation of the form

$$f(\lambda(\nabla^2 u + \chi)) = e^{u_t + \psi}. \tag{1.15}$$

Applying the methods of [9], Bao and Dong [3] solved (1.1)-(1.2) under an additional condition which is introduced in [9] (see [11] also)

$$T_\lambda \cap \partial\Gamma^\sigma \text{ is a nonempty compact set}, \forall \lambda \in \Gamma \text{ and } \sup_{\partial\Gamma} f < \sigma < f(\lambda), \tag{1.16}$$

where $\partial\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) = \sigma\}$ is the boundary of $\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}$ and T_λ denote the tangent plane at λ of $\partial\Gamma^{f(\lambda)}$, for $\sigma > \sup_{\partial\Gamma} f$ and $\lambda \in \Gamma$. The reader is referred to [19], [27], [8], [9], [10], [11], [12] and their references for the study of elliptic Hessian equations on manifolds.

We can prove the short time existence as Theorem 15.9 in [21]. So without of loss of generality, we may assume

$$f(\lambda(\nabla^2\varphi(x,0)+\chi(x)),-\varphi_t(x,0))=\psi(x,0) \text{ for all } x\in\overline{M}. \tag{1.17}$$

As usual, the main part of this paper is to derive the *a priori* C^2 estimates. We see that (1.1) is uniformly parabolic after establishing the C^2 estimates by (1.3) and (1.5). The $C^{2,\alpha}$ estimates can be obtained by applying Evans-Krylov theorem (see [5] and [18]). Finally Theorem 1.1 can be proved as Theorem 15.9 of [21].

The rest of this paper is organized as follows. In section 2, we introduce some notations and useful lemmas. C^1 estimates are derived in Section 3. An *a priori* bound for $|u_t|$ is obtained in Section 4. Section 5 and Section 6 are devoted to the global and boundary estimates for second order derivatives respectively.

2. Preliminaries

Let F be the function defined by $F(A,\tau)=f(\lambda(A),\tau)$ for $A\in\mathbb{S}^n$, $\tau\in\mathbb{R}$ with $(\lambda(A),\tau)\in\Gamma$, where $\mathbb{S}^n$ is the set of $n\times n$ symmetric matrices. It was shown in [4] that F is concave from (1.4). For simplicity we shall use the notations $U=\nabla^2u+\chi$, $\underline{U}=\nabla^2\underline{u}+\chi$ and under an orthonormal local frame $e_1,\dots,e_n$,

$$U_{ij}\equiv U(e_i,e_j)=\nabla_{ij}u+\chi_{ij},\;\; \underline{U}_{ij}\equiv \underline{U}(e_i,e_j)=\nabla_{ij}\underline{u}+\chi_{ij}.$$

Thus, (1.1) can be written in the form locally

$$F(U,-u_t)=f(\lambda(U_{ij}),-u_t)=\psi. \tag{2.1}$$

Let

$$F^{ij}=\frac{\partial F}{\partial A_{ij}}(U,-u_t), F^{\tau}=\frac{\partial F}{\partial \tau}(U,-u_t)$$
$$F^{ij,kl}=\frac{\partial^2 F}{\partial A_{ij}\partial A_{kl}}(U,-u_t), F^{ij,\tau}=\frac{\partial^2 F}{\partial A_{ij}\partial \tau}(U,-u_t).$$

By (1.3) we see that $F^\tau>0$ and $\{F^{ij}\}$ is positive definite. We shall also denote the eigenvalues of $\{F^{ij}\}$ by $f_1,\dots,f_n$ when there is no possible confusion. We note that $\{U_{ij}\}$ and $\{F^{ij}\}$ can be diagnolized simultaneously and that

$$F^{ij}U_{ij}=\sum f_i\widehat{\lambda}_i,\;\; F^{ij}U_{ik}U_{kj}=\sum f_i\widehat{\lambda}_i^2,$$

where $\lambda(\{U_{ij}\})=(\widehat{\lambda}_1,\dots,\widehat{\lambda}_n)$.

Similarly to [10], we write

$$\begin{aligned}\mu(x,t)=&(\lambda(\underline{U}(x,t)),-\underline{u}_t(x,t)),\\ \lambda(x,t)=&(\lambda(U(x,t)),-u_t(x,t))\end{aligned}$$

and $\nu_\lambda\equiv Df(\lambda)/|Df(\lambda)|$ is the unit normal vector to the level hypersurface $\partial\Gamma^{f(\lambda)}$ for $\lambda\in\Gamma$. Since $K\equiv\{\mu(x,t):(x,t)\in\overline{M_T}\}$ is a compact subset of Γ, there exist uniform constants $\beta\in(0,\frac{1}{2\sqrt{n+1}})$ such that

$$\nu_{\mu(x,t)}-2\beta\mathbf{1}\in\Gamma_{n+1},\forall(x,t)\in\overline{M_T}. \tag{2.2}$$

where $\mathbf{1}=(1,\cdots,1)\in\mathbb{R}^{n+1}$ (see [10]).

We need the following Lemma which is proved in [10].

Lemma 2.1. *Suppose that* $|\nu_\mu - \nu_\lambda| \geq \beta$. *Then there exists a uniform constant* $\varepsilon > 0$ *such that*

$$\sum_{i=1}^{n+1} f_i(\lambda)(\mu_i - \lambda_i) \geq \varepsilon\Big(1 + \sum_{i=1}^{n+1} f_i(\lambda)\Big). \tag{2.3}$$

Define the linear operator $\mathcal{L}$ locally by

$$\mathcal{L}v = F^{ij}\nabla_{ij}v - F^\tau v_t, \text{ for } v \in C^2(\overline{M_T}).$$

From Lemma 2.1 and Lemma 6.2 of [4] it is easy to derive that when $|\nu_{\mu(x,t)} - \nu_{\lambda(x,t)}| \geq \beta$,

$$\mathcal{L}(\underline{u} - u) \geq \varepsilon\Big(1 + \sum F^{ii} + F^\tau\Big). \tag{2.4}$$

If $|\nu_\mu - \nu_\lambda| < \beta$, we have $\nu_\lambda - \beta\mathbf{1} \in \Gamma_{n+1}$. It follows that

$$f_i \geq \frac{\beta}{\sqrt{n+1}} \sum_{j=1}^{n+1} f_j, \ \forall 1 \leq i \leq n+1. \tag{2.5}$$

3. The C^1 estimates

Since u is admissible and $\Gamma \subset \{\lambda \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \lambda_i > 0\}$, we see that u is a subsolution of

$$\begin{cases} \triangle h - h_t + \mathrm{tr}(\chi) = 0, & \text{in } M_T, \\ h = \varphi, & \text{on } \mathcal{P}M_T. \end{cases} \tag{3.1}$$

Let h be the solution of (3.1). It follows from the maximum principle that $\underline{u} \leq u \leq h$ on $\overline{M_T}$. Therefore, we have

$$\sup_{\overline{M_T}} |u| + \sup_{\mathcal{P}M_T} |\nabla u| \leq C. \tag{3.2}$$

For the global gradient estimates, we can prove the following theorem.

Theorem 3.1. *Suppose that* (1.3), (1.4), (1.9) *and* (1.12) *hold. Let* $u \in C^3(\overline{M_T})$ *be an admissible solution of (1.1) in* M_T. *Then*

$$\sup_{\overline{M_T}} |\nabla u| \leq C(1 + \sup_{\mathcal{P}M_T} |\nabla u|), \tag{3.3}$$

where C depends on $|\psi|_{C^1(\overline{M_T})}$, $|u|_{C^0(\overline{M_T})}$ *and other known data.*

Proof. Set

$$W = \sup_{(x,t)\in\overline{M_T}} we^\phi,$$

where $w = \frac{|\nabla u|^2}{2}$ and ϕ is a function to be determined. It suffices to estimate W and we may assume that W is achieved at $(x_0, t_0) \in \overline{M_T} - \mathcal{P}M_T$. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about x_0 such that $\nabla_{e_i} e_j = 0$ at x_0 and $U(x_0, t_0)$ is diagonal. We see that the function $\log w + \phi$ attains its maximum at (x_0, t_0). Therefore, at (x_0, t_0), we have

$$\frac{\nabla_i w}{w} + \nabla_i \phi = 0, \ \text{ for each } i = 1, \ldots, n, \tag{3.4}$$

$$\frac{w_t}{w} + \phi_t \geq 0 \tag{3.5}$$

and

$$\frac{\nabla_{ii} w}{w} - \Big(\frac{\nabla_i w}{w}\Big)^2 + \nabla_{ii}\phi \leq 0. \tag{3.6}$$

Differentiating the equation (1.1), we get

$$F^{ii}\nabla_k U_{ii} - F^\tau \nabla_k u_t = \nabla_k \psi \text{ for } k = 1, \ldots, n \tag{3.7}$$

and

$$F^{ii}(U_{ii})_t - F^\tau u_{tt} = \psi_t. \tag{3.8}$$

Note that

$$\nabla_i w = \nabla_k u \nabla_{ik} u, \ w_t = \nabla_k u (\nabla_k u)_t, \ \nabla_{ii} w = (\nabla_{ik} u)^2 + \nabla_k u \nabla_{iik} u \tag{3.9}$$

and that

$$\nabla_{ijk} u - \nabla_{jik} u = R^l_{kij} \nabla_l u. \tag{3.10}$$

We have, by (3.5), (3.7), (3.9) and (3.10),

$$\begin{aligned} F^{ii}\nabla_{ii} w \geq & \nabla_k u F^{ii} \nabla_{iik} u \\ \geq & -C|\nabla u| - C|\nabla u|^2 \sum F^{ii} + F^\tau \nabla_k u \nabla_k u_t \\ \geq & -C|\nabla u| - C|\nabla u|^2 \sum F^{ii} - w F^\tau \phi_t, \end{aligned} \tag{3.11}$$

provided $|\nabla u|$ is sufficiently large. Combining (3.4), (3.6), (3.11), we obtain

$$0 \geq -\frac{C}{|\nabla u|} - C \sum F^{ii} - F^{ii}(\nabla_i \phi)^2 + \mathcal{L}\phi. \tag{3.12}$$

Let $\phi = \delta v^2$, where $v = u + \sup_{\overline{M_T}} |u| + 1$ and δ is a small positive constant to be chosen. Thus, choosing δ sufficiently small such that $2\delta - 4\delta^2 v^2 \geq c_0 > 0$ for some uniform constant c_0, by (1.12),

$$\begin{aligned} \mathcal{L}\phi - F^{ii}(\nabla_i\phi)^2 = & 2\delta v (F^{ii}\nabla_{ii} u - F^\tau u_t) + (2\delta - 4\delta^2 v^2) F^{ii} (\nabla_i u)^2 \\ \geq & -C\delta\Big(1 + \sum F^{ii} + F^\tau\Big) + c_0 F^{ii}(\nabla_i u)^2. \end{aligned} \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$c_0 F^{ii}(\nabla_i u)^2 \leq C\Big(1 + \sum F^{ii} + F^\tau\Big), \tag{3.14}$$

provided $|\nabla u|$ is sufficiently large. We may assume $|\nabla u(x_0, t_0)| \leq n \nabla_1 u(x_0, t_0)$ and by (3.4),

$$U_{11} = -2\delta v w + \frac{\nabla_k u \chi_{1k}}{\nabla_1 u} < 0$$

provided w is sufficiently large. Then we can derive from (1.9) that

$$F^{11} \geq \nu_0 \Big(1 + \sum F^{ii} + F^\tau\Big).$$

Therefore, we obtain a bound $|\nabla u(x_0, t_0)| \leq Cn^2/c_0\nu_0$ by (3.14) and (3.3) is proved. □

Remark 3.2. We see that in the proof of Theorem 3.1, we do not need the existence of $\underline{u}$.

By (3.2) and (3.3), the C^1 estimates are established.

4. ESTIMATE FOR $|u_t|$

In this section, we derive the estimate for $|u_t|$.

Theorem 4.1. *Suppose that* (1.3), (1.4), (1.10) *and* (1.11) *hold. Let* $u \in C^3(\overline{M_T})$ *be an admissible solution of (1.1) in* M_T. *Then there exists a positive constant* C *depending on* $|u|_{C^1(\overline{M_T})}$, $|\underline{u}|_{C^2(\overline{M_T})}$, $|\psi|_{C^2(\overline{M_T})}$ *and other known data such that*

$$\sup_{\overline{M_T}} |u_t| \leq C(1 + \sup_{\mathcal{P}M_T} |u_t|). \tag{4.1}$$

Proof. We first show that

$$\sup_{\overline{M_T}} (-u_t) \leq C(1 + \sup_{\mathcal{P}M_T} (-u_t)) \tag{4.2}$$

for which we set

$$W = \sup_{\overline{M_T}} (-u_t) e^{\phi},$$

where ϕ is a function to be chosen. We may assume that W is attained at $(x_0, t_0) \in \overline{M_T} - \mathcal{P}M_T$. As in the proof of Theorem 3.1, we choose an orthonormal local frame $e_1, \cdots, e_n$ about x_0 such that $\nabla_{e_i} e_j = 0$ and $\{U_{ij}(x_0, t_0)\}$ is diagonal. We may assume $-u_t(x_0, t_0) > 0$. At (x_0, t_0) where the function $\log(-u_t) + \phi$ achieves its maximum, we have

$$\frac{\nabla_i u_t}{u_t} + \nabla_i \phi = 0, \text{ for each } i = 1, \ldots, n, \tag{4.3}$$

$$\frac{u_{tt}}{u_t} + \phi_t \geq 0, \tag{4.4}$$

and

$$0 \geq F^{ii} \Big\{ \frac{\nabla_{ii} u_t}{u_t} - \Big(\frac{\nabla_i u_t}{u_t} \Big)^2 + \nabla_{ii} \phi \Big\}. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we find

$$0 \geq \frac{1}{u_t} (F^{ii} \nabla_{ii} u_t - F^{\tau} u_{tt}) - F^{ii} (\nabla_i \phi)^2 + \mathcal{L}\phi. \tag{4.6}$$

By (3.8) and (4.6),

$$\mathcal{L}\phi \leq -\frac{\psi_t}{u_t} + F^{ii} (\nabla_i \phi)^2. \tag{4.7}$$

Let $\phi = \frac{\delta^2}{2} |\nabla u|^2 + \delta u + b(\underline{u} - u)$, where $\delta \ll b \ll 1$ are positive constants to be determined. By straightforward calculations, we see

$$\begin{aligned}
\nabla_i \phi =& \delta^2 \nabla_k u \nabla_{ik} u + \delta \nabla_i u + b \nabla_i (\underline{u} - u), \\
\phi_t =& \delta^2 \nabla_k u (\nabla_k u)_t + \delta u_t + b(\underline{u} - u)_t, \\
\nabla_{ii} \phi =& \delta^2 (\nabla_{ik} u)^2 + \delta^2 \nabla_k u \nabla_{iik} u + \delta \nabla_{ii} u + b \nabla_{ii} (\underline{u} - u).
\end{aligned}$$

It follows that, in view of (3.7) and (3.10),

$$\begin{aligned}
\mathcal{L}\phi \geq & \, \delta^2 \nabla_k u (F^{ii} \nabla_{iik} u - F^{\tau} (\nabla_k u)_t) + \frac{\delta^2}{2} F^{ii} U_{ii}^2 \\
& + \delta F^{ii} \nabla_{ii} u - \delta F^{\tau} u_t - C\delta^2 \sum F^{ii} + b\mathcal{L}(\underline{u} - u) \\
\geq & - C\delta^2 \Big(1 + \sum F^{ii} \Big) + \frac{\delta^2}{2} F^{ii} U_{ii}^2 + \delta \mathcal{L} u + b\mathcal{L}(\underline{u} - u).
\end{aligned} \tag{4.8}$$

Next,

$$(\nabla_i \phi)^2 \leq C\delta^4 U_{ii}^2 + Cb^2. \tag{4.9}$$

Thus, we can derive from (4.7), (4.8) and (4.9) that

$$b\mathcal{L}(\underline{u} - u) + \frac{\delta^2}{4} F^{ii} U_{ii}^2 + \delta \mathcal{L} u \leq -\frac{C}{u_t} + C\delta^2 \Big(1 + \sum F^{ii}\Big) + Cb^2 \sum F^{ii}, \tag{4.10}$$

when δ is small enough. Now we use the idea of [10] to consider two cases: (i) $|\nu_{\mu_0} - \nu_{\lambda_0}| \geq \beta$ and (ii) $|\nu_{\mu_0} - \nu_{\lambda_0}| < \beta$, where $\mu_0 = \mu(x_0, t_0)$ and $\lambda_0 = \lambda(x_0, t_0)$.

In case (i), by Lemma 2.1, we see that (2.4) holds. By (1.10), we have

$$\mathcal{L} u \geq F^{ii} U_{ii} - F^\tau u_t - C \sum F^{ii} \geq -K_0 \Big(\sum F^{ii} + F^\tau\Big) - C \sum F^{ii}. \tag{4.11}$$

Combining with (4.11) and (4.10), we have

$$b\mathcal{L}(\underline{u} - u) \leq -\frac{C}{u_t} + C\delta \Big(1 + \sum F^{ii} + F^\tau\Big) + Cb^2 \sum F^{ii}. \tag{4.12}$$

Now using (2.4) we can choose $\delta \ll b \ll 1$ to obtain a bound $-u_t(x_0, t_0) \leq \frac{C}{b\varepsilon}$.

In case (ii), we see that (2.5) holds. By (4.10), we have

$$\begin{aligned} b\mathcal{L}(\underline{u} - u) + &\frac{\delta^2}{4} F^{ii} U_{ii}^2 + \delta (F^{ii} U_{ii} - F^\tau u_t) \\ &\leq -\frac{C}{u_t} + C\delta^2 \Big(1 + \sum F^{ii}\Big) + C(\delta + b^2) \sum F^{ii}. \end{aligned} \tag{4.13}$$

Note that

$$\frac{\delta^2}{4} F^{ii} U_{ii}^2 \geq \delta F^{ii} |U_{ii}| - \sum F^{ii} \tag{4.14}$$

and

$$\mathcal{L}(\underline{u} - u) \geq 0 \tag{4.15}$$

by the concavity of F. Therefore, by (4.13), (4.14) and (4.15), we have

$$-\delta F^\tau u_t \leq -\frac{C}{u_t} + C\delta^2 + C \sum F^{ii}. \tag{4.16}$$

By (1.10), similar to [10],

$$\begin{aligned} -u_t \Big(\sum F^{ii} + F^\tau\Big) &\geq f(-u_t \mathbf{1}) - f(\lambda(U), -u_t) + \sum F^{ii} U_{ii} - F^\tau u_t \\ &\geq f(-u_t \mathbf{1}) - f(\lambda(\underline{U}), -\underline{u}_t) - K_0 \Big(\sum F^{ii} + F^\tau\Big) \\ &\geq 2b_0 + u_t \Big(\sum F^{ii} + F^\tau\Big) \end{aligned} \tag{4.17}$$

for some uniform constant $b_0 > 0$, provided $-u_t$ is sufficiently large, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{n+1}$. It follows that, by (2.5),

$$-F^\tau u_t \geq \frac{-\beta u_t}{\sqrt{n+1}} \Big(\sum F^{ii} + F^\tau\Big) \geq \frac{-\beta u_t}{2\sqrt{n+1}} \Big(\sum F^{ii} + F^\tau\Big) + \frac{\beta b_0}{2\sqrt{n+1}}.$$

Choose δ sufficiently small such that

$$\frac{\beta b_0 \delta}{2\sqrt{n+1}} - C\delta^2 \geq c_1 > 0$$

for some constant c_1. Therefore, we can derive from (4.16) that

$$-u_t(x_0,t_0) \le \max\Big\{\frac{C\sqrt{n+1}}{\beta\delta}, \frac{C}{c_1}\Big\}.$$

So (4.2) holds.

Similarly, we can show

$$\sup_{\overline{M_T}} u_t \le C(1+\sup_{\mathcal{P}M_T} u_t) \tag{4.18}$$

by setting

$$W = \sup_{\bar{M}_T} u_t e^{\phi}$$

and $\phi = \frac{\delta^2}{2}|\nabla u|^2 - \delta u + b(\underline{u} - u)$.

Combining (4.2) and (4.18), we can see that (4.1) holds. □

Since $u_t = \varphi_t$ on SM_T and (1.17), we can derive the estimate

$$\sup_{\overline{M_T}} |u_t| \le C. \tag{4.19}$$

5. Global estimates for second order derivatives

In this section, we derive the global estimates for the second order derivatives. We prove the following maximum principle.

Theorem 5.1. *Let $u \in C^4(\overline{M_T})$ be an admissible solution of (1.1) in M_T. Suppose that* (1.3)*,* (1.4) *and* (1.11) *hold. Then*

$$\sup_{\overline{M_T}} |\nabla^2 u| \le C(1+\sup_{\mathcal{P}M_T} |\nabla^2 u|), \tag{5.1}$$

where $C>0$ depends on $|u|_{C^1(\overline{M_T})}$, $|u_t|_{C^0(\overline{M_T}))}$, $|\psi|_{C^2(\overline{M_T}))}$ and other known data.

Proof. Set

$$W = \max_{(x,t)\in\overline{M_T}} \max_{\xi\in T_xM, |\xi|=1} (\nabla_{\xi\xi} u + \chi(\xi,\xi))e^{\phi},$$

where ϕ is a function to be determined. We may assume W is achieved at $(x_0,t_0) \in \overline{M_T} - \mathcal{P}M_T$ and $\xi_0 \in T_{x_0}M$. Choose a smooth orthonormal local frame $e_1,\dots,e_n$ about x_0 as before such that $\xi_0 = e_1$, $\nabla_{e_i} e_j = 0$, and $\{U_{ij}(x_0,t_0)\}$ is diagonal. We see that $W = U_{11}(x_0,t_0)e^{\phi(x_0,t_0)}$. We may also assume that $U_{11} \ge \ldots \ge U_{nn}$ at (x_0,t_0).

Since the function $\log(U_{11}) + \phi$ attains its maximum at (x_0,t_0), we have, at (x_0,t_0),

$$\frac{\nabla_i U_{11}}{U_{11}} + \nabla_i\phi = 0 \text{ for each } i = 1,\dots,n, \tag{5.2}$$

$$\frac{(\nabla_{11}u)_t}{U_{11}} + \phi_t \ge 0, \tag{5.3}$$

and

$$0 \ge \sum_i F^{ii}\Big\{\frac{\nabla_{ii}U_{11}}{U_{11}} - \Big(\frac{\nabla_i U_{11}}{U_{11}}\Big)^2 + \nabla_{ii}\phi\Big\}. \tag{5.4}$$

Therefore, by (5.3) and (5.4), we find

$$\mathcal{L}\phi \leq -\frac{1}{U_{11}}(F^{ii}\nabla_{ii}U_{11} - F^{\tau}(\nabla_{11}u)_t) + F^{ii}\Big(\frac{\nabla_i U_{11}}{U_{11}}\Big)^2. \tag{5.5}$$

By the formula

$$\begin{aligned} \nabla_{ijkl}v - \nabla_{klij}v =& R^m_{ljk}\nabla_{im}v + \nabla_i R^m_{ljk}\nabla_m v + R^m_{lik}\nabla_{jm}v \\ &+ R^m_{jik}\nabla_{lm}v + R^m_{jil}\nabla_{km}v + \nabla_k R^m_{jil}\nabla_m v \end{aligned} \tag{5.6}$$

we have

$$\nabla_{ii}U_{11} \geq \nabla_{11}U_{ii} - CU_{11}, \tag{5.7}$$

Differentiating equation (1.1) twice, we have

$$\begin{aligned} &F^{ij}\nabla_{11}U_{ij} - F^{\tau}\nabla_{11}u_t + F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} \\ &+F^{\tau\tau}(\nabla_1 u_t)^2 - 2F^{ij,\tau}\nabla_1 U_{ij}\nabla_1 u_t = \nabla_{11}\psi \geq -C. \end{aligned} \tag{5.8}$$

It follows from (5.5), (5.7) and (5.8) that

$$\mathcal{L}\phi \leq \frac{C}{U_{11}} + C\sum F^{ii} + E, \tag{5.9}$$

where

$$E = \frac{1}{U_{11}}\Big(F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} - 2F^{ij,\tau}\nabla_1 U_{ij}\nabla_1 u_t + F^{\tau\tau}(\nabla_1 u_t)^2\Big) + F^{ii}\Big(\frac{\nabla_i U_{11}}{U_{11}}\Big)^2.$$

E can be estimated as in [9] using an idea of Urbas [27] to which the following inequality proved by Andrews [1] and Gerhardt [7] is crucial.

Lemma 5.2. *For any symmetric matrix $\eta = \{\eta_{ij}\}$ we have*

$$F^{ij,kl}\eta_{ij}\eta_{kl} = \sum_{i,j}\frac{\partial^2 f}{\partial\lambda_i\partial\lambda_j}\eta_{ii}\eta_{jj} + \sum_{i\neq j}\frac{f_i - f_j}{\lambda_i - \lambda_j}\eta_{ij}^2.$$

The second term on the right hand side is nonpositive if f is concave, and is interpreted as a limit if $\lambda_i = \lambda_j$.

Similar to [9], we can derive (see [3] also)

$$E \leq \sum_{i\in J}F^{ii}(\nabla_i\phi)^2 + C\sum_{i\in K}F^{ii} + CF^{11}\sum_{i\in K}(\nabla_i\phi)^2, \tag{5.10}$$

where $J = \{i : 3U_{ii} \leq -U_{11}\}$ and $K = \{i : 3U_{ii} > -U_{11}\}$.

Let

$$\phi = \frac{\delta|\nabla u|^2}{2} + b(\underline{u} - u),$$

where δ and b are positive constants to be determined. Thus, we can derive from (5.10) that

$$E \leq Cb^2\sum_{i\in J}F^{ii} + C\delta^2 F^{ii}U_{ii}^2 + C\sum_{i\in K}F^{ii} + C(\delta^2 U_{11}^2 + b^2)F^{11}. \tag{5.11}$$

On the other hand, by (3.7) and (3.10),

$$\begin{aligned} \mathcal{L}\phi =& \delta F^{ii}\sum_k(\nabla_{ik}u)^2 + \delta\nabla_k u F^{ii}\nabla_{iik}u - \delta\nabla_k u F^{\tau}(\nabla_k u)_t + b\mathcal{L}(\underline{u} - u) \\ \geq& \delta F^{ii}U_{ii}^2 + b\mathcal{L}(\underline{u} - u) - C\delta\Big(1 + \sum F^{ii}\Big) \end{aligned} \tag{5.12}$$

Combining (5.9), (5.11) and (5.12), we obtain

$$\frac{\delta}{2}F^{ii}U_{ii}^2 + b\mathcal{L}(\underline{u}-u) \leq \frac{C}{U_{11}} + Cb^2\sum_{i\in J}F^{ii} + Cb^2F^{11} + C\Big(1+\sum F^{ii}\Big) \tag{5.13}$$

provided δ is sufficiently small. Note that $|U_{jj}| \geq \frac{1}{3}U_{11}$, for $j \in J$. Therefore, by (5.13), we have

$$\frac{\delta}{4}F^{ii}U_{ii}^2 + b\mathcal{L}(\underline{u}-u) \leq C\Big(1+\sum F^{ii}\Big) \tag{5.14}$$

when $U_{11}^2 \geq \max\{Cb^2/\delta, 1\}$.

Now let $\mu_0 = \mu(x_0,t_0)$ and $\lambda_0 = \lambda(x_0,t_0)$. If $|\lambda_0 - \mu_0| \geq \beta$, we can obtain a bound of $U_{11}(x_0,t_0)$ by (2.4) as in [9].

If $|\lambda_0 - \mu_0| < \beta$, we see that (2.5) holds. Let $\widehat{\lambda} = \lambda(U(x_0,t_0))$. We may assume $|\widehat{\lambda}| \geq |u_t(x_0,t_0)|$. Similar to [10], by the concavity of f,

$$\begin{aligned} |\widehat{\lambda}|\Big(\sum F^{ii} + F^\tau\Big) &\geq f(|\widehat{\lambda}|\mathbf{1}) - f(\lambda(U), -u_t) + \sum F^{ii}U_{ii} - F^\tau u_t \\ &\geq f(|\widehat{\lambda}|\mathbf{1}) - f(\lambda(\underline{U}), -\underline{u}_t) - |\widehat{\lambda}|\Big(\sum F^{ii} + F^\tau\Big) \\ &\geq 2b_0 - |\widehat{\lambda}|\Big(\sum F^{ii} + F^\tau\Big) \end{aligned} \tag{5.15}$$

for some uniform positive constant b_0, provided $|\widehat{\lambda}|$ is sufficiently large. By (2.5), (4.15) and (5.14), we see that

$$2c_0|\widehat{\lambda}|^2\Big(\sum F^{ii} + F^\tau\Big) \leq C\Big(1+\sum F^{ii}\Big), \tag{5.16}$$

where

$$c_0 := \frac{\delta\beta}{8\sqrt{n+1}}.$$

Then we can derive a bound of $|\widehat{\lambda}|$ from (5.15). □

6. Boundary estimates for second order derivatives

In this section, we consider the estimates of second order derivatives on SM_T. We may assume $\varphi \in C^4(\overline{M_T})$. For simplicity we shall make use of the condition (1.12) though stronger results may be proved (see [9], [10] and [12]).

The pure tangential second derivatives are easy to estimate from the boundary condition $u = \varphi$ on $\mathcal{P}M_T$. So we are focused on the estimates for mixed tangential-normal and pure normal second derivatives.

Fix a point $(x_0,t_0) \in SM_T$. We shall choose smooth orthonormal local frames $e_1,\ldots,e_n$ around x_0 such that when restricted to ∂M, e_n is normal to ∂M.

Let $\rho(x)$ and $d(x)$ denote the distance from $x \in M$ to x_0 and ∂M respectively and set

$$M_T^\delta = \{X = (x,t) \in M \times (0,T] : \rho(x) < \delta\}.$$

We shall use the following barrier function as in [9].

$$\Psi = A_1 v + A_2\rho^2 - A_3\sum_{\gamma<n}|\nabla_\gamma(u-\varphi)|^2, \tag{6.1}$$

where

$$v = (u-\underline{u}) + ad - \frac{Nd^2}{2}.$$

Now we show the following lemma which is useful to construct barrier functions (see Lemma 6.2).

Lemma 6.1. *Suppose* (1.4) *and* (1.12) *hold. Then for any* $\sigma > 0$ *and any index* r*,*

$$\sum f_i|\widehat{\lambda}_i| \leq \sigma \sum_{i\neq r} f_i\widehat{\lambda}_i^2 + \frac{C}{\sigma}\Big(\sum f_i + F^\tau\Big) + C \tag{6.2}$$

Proof. If $\widehat{\lambda}_r < 0$, by (1.12), we see

$$\begin{aligned}\sum f_i|\widehat{\lambda}_i| &= 2\sum_{\widehat{\lambda}_i>0} f_i\widehat{\lambda}_i - \sum f_i\widehat{\lambda}_i + F^\tau u_t - F^\tau u_t \\ &\leq \sigma \sum_{\widehat{\lambda}_i>0} f_i\widehat{\lambda}_i^2 + \frac{1}{\sigma}\sum_{\widehat{\lambda}_i>0} f_i + K_0\Big(1 + \sum f_i + F^\tau\Big) + CF^\tau\end{aligned}$$

and (6.2) follows.

If $\widehat{\lambda}_r \geq 0$, by the concavity of f,

$$\begin{aligned}\sum f_i|\widehat{\lambda}_i| = \sum f_i\widehat{\lambda}_i - 2\sum_{\widehat{\lambda}_i<0} f_i\widehat{\lambda}_i \\ \leq \sigma \sum_{\widehat{\lambda}_i<0} f_i\widehat{\lambda}_i^2 + \frac{1}{\sigma}\sum_{\widehat{\lambda}_i<0} f_i + \sum f_i\widehat{\mu}_i - F^\tau(\underline{u} - u)_t \\ \leq \sigma \sum_{\widehat{\lambda}_i<0} f_i\widehat{\lambda}_i^2 + \frac{1}{\sigma}\sum_{\widehat{\lambda}_i<0} f_i + C\Big(\sum f_i + F^\tau\Big).\end{aligned}$$

Then (6.2) is proved. □

The following Lemma is crucial to our estimates and the idea is mainly from [10] and [11] (see [13] also).

Lemma 6.2. *Suppose that* (1.3)*,* (1.4) *and* (1.11) *hold. Then for any constant* $K > 0$*, there exist uniform positive constants* a, δ *sufficiently small, and* A_1, A_2, A_3, N *sufficiently large such that* $\Psi \geq K(d + \rho^2)$ *in* $\overline{M^\delta_T}$ *and*

$$\mathcal{L}\Psi \leq -K\Big(1 + \sum_{i=1}^n f_i|\widehat{\lambda}_i| + \sum_{i=1}^n f_i + F^\tau\Big) \text{ in } M^\delta_T. \tag{6.3}$$

Proof. For any fixed $(x, t) \in M^\delta_T$, we may assume that U_{ij} and F^{ij} are both diagonal at (x, t). Firstly, we have (see [9] for details),

$$\mathcal{L}(\nabla_k(u - \varphi)) \leq C\Big(1 + \sum f_i|\widehat{\lambda}_i| + \sum f_i + F^\tau\Big), \ \forall\, 1 \leq k \leq n. \tag{6.4}$$

Therefore,

$$\sum_{l<n}\mathcal{L}(|\nabla_l(u - \varphi)|^2) \geq \sum_{l<n} F^{ij}U_{il}U_{jl} - C\Big(1 + \sum f_i|\widehat{\lambda}_i| + \sum f_i + F^\tau\Big). \tag{6.5}$$

Using the same proof of Proposition 2.19 in [9], we can show

$$\sum_{l<n} F^{ij}U_{il}U_{jl} \geq \frac{1}{2}\sum_{i\neq r} f_i\widehat{\lambda}_i^2, \tag{6.6}$$

for some index r. Write $\mu = \mu(x, t)$ and $\lambda = \lambda(x, t)$ and note that $\mu = (\widehat{\mu}, -\underline{u}_t)$ and $\lambda = (\widehat{\lambda}, -u_t)$, where $\widehat{\mu} = \lambda(\underline{U})$.

We shall consider two cases as before: (**a**) $|\nu_\mu - \nu_\lambda| < \beta$ and (**b**) $|\nu_\mu - \nu_\lambda| \geq \beta$.

Case **(a)**. By (2.5), we have

$$f_i \geq \frac{\beta}{\sqrt{n+1}}\Big(\sum f_k + F^\tau\Big), \quad \forall\, 1 \leq i \leq n. \tag{6.7}$$

Now we make a little modification of the proof of Lemma 3.1 in [10] to show the following inequality

$$\sum_{i\neq r} f_i \widehat{\lambda}_i^2 \geq c_0 \sum f_i \widehat{\lambda}_i^2 - C_0\Big(\sum f_i + F^\tau\Big) \tag{6.8}$$

for some $c_0, C_0 > 0$. If $\widehat{\lambda}_r < 0$, we have

$$\widehat{\lambda}_r^2 \leq n \sum_{i\neq r} \widehat{\lambda}_i^2 + C, \tag{6.9}$$

where C depends on the bound of u_t since

$$\sum \widehat{\lambda}_i - u_t > 0.$$

Therefore, by (6.7) and (6.9), we have

$$f_r \widehat{\lambda}_r^2 \leq n f_r \sum_{i\neq r} \widehat{\lambda}_i^2 + C f_r \leq \frac{n\sqrt{n+1}}{\beta} \sum_{i\neq r} f_i \widehat{\lambda}_i^2 + C \sum f_i \tag{6.10}$$

and (6.8) holds.

Now suppose $\widehat{\lambda}_r \geq 0$. By the concavity of f,

$$f_r \widehat{\lambda}_r \leq f_r \widehat{\mu}_r - F^\tau(\underline{u}_t - u_t) + \sum_{i\neq r} f_i(\widehat{\mu}_i - \widehat{\lambda}_i). \tag{6.11}$$

Thus, by (6.7) and Schwarz inequality, we have

$$\begin{aligned}
&\frac{\beta f_r \widehat{\lambda}_r^2}{\sqrt{n+1}}\Big(\sum f_i + F^\tau\Big) \\
&\leq f_r^2 \widehat{\lambda}_r^2 \leq C\Big(f_r^2 \widehat{\mu}_r^2 + \sum_{k\neq r} f_k \sum_{i\neq r} f_i(\widehat{\mu}_i^2 + \widehat{\lambda}_i^2) + (F^\tau)^2\Big) \\
&\leq C\Big(\sum f_i + F^\tau\Big)\Big\{\Big(\sum f_i + F^\tau\Big) + \sum_{i\neq r} f_i \widehat{\lambda}_i^2\Big\},
\end{aligned} \tag{6.12}$$

where C may depend on the bound of $|u_t|$. It follows that

$$f_r \widehat{\lambda}_r^2 \leq C \sum_{i\neq r} f_i \widehat{\lambda}_i^2 + C\Big(\sum f_i + F^\tau\Big) \tag{6.13}$$

and (6.8) holds.

We first suppose $|\lambda| \geq R$ for R sufficiently large. By (5.15), we see

$$\sum f_i \widehat{\lambda}_i^2 \geq b_0 |\widehat{\lambda}| \tag{6.14}$$

when R is sufficiently large. Since $|\nabla d| \equiv 1$, when a and δ are sufficiently small, by (6.7), we have,

$$\begin{aligned}
\mathcal{L}v &\leq \Big(\mathcal{L}(u - \underline{u}) + C_0(a + Nd)\sum f_i - N F^{ij} \nabla_i d \nabla_j d\Big) \\
&\leq -\frac{\beta N}{2\sqrt{n+1}}\Big(\sum f_k + F^\tau\Big).
\end{aligned} \tag{6.15}$$

Note that for any $\sigma > 0$,

$$\sum f_i|\widehat{\lambda}_i| \leq \sigma \sum f_i \widehat{\lambda}_i^2 + \frac{1}{\sigma}\sum f_i. \tag{6.16}$$

Therefore, it follows from (6.8), (6.15) and (6.16) that for any $\sigma > 0$,

$$\begin{aligned}
\mathcal{L}\Psi \leq & -\frac{\beta A_1 N}{2\sqrt{n+1}}\Big(\sum f_k + F^\tau\Big) + CA_2 \sum f_i \\
& - \frac{A_3}{2}\sum_{i\neq r} f_i \widehat{\lambda}_i^2 + CA_3\Big(1 + \sum f_i|\widehat{\lambda}_i| + \sum f_i + F^\tau\Big) \\
\leq & -\frac{\beta A_1 N}{2\sqrt{n+1}}\Big(\sum f_k + F^\tau\Big) - \frac{A_3 c_0}{2}\sum f_i \widehat{\lambda}_i^2 + CA_2 \sum f_i \\
& + CA_3\Big(1 + \sum f_i|\widehat{\lambda}_i| + \sum f_i + F^\tau\Big) \\
\leq & -\frac{\beta A_1 N}{2\sqrt{n+1}}\Big(\sum f_k + F^\tau\Big) + \Big(A_3\sigma - \frac{A_3 c_0}{2}\Big)\sum f_i \widehat{\lambda}_i^2 \\
& + C\Big(A_2 + \frac{A_3}{\sigma}\Big)\sum f_i + CA_3\Big(1 + F^\tau\Big).
\end{aligned} \tag{6.17}$$

Let $\sigma = c_0/4$, we find

$$\begin{aligned}
\mathcal{L}\Psi \leq & -\frac{\beta A_1 N}{2\sqrt{n+1}}\Big(\sum f_k + F^\tau\Big) - \frac{A_3 c_0}{4}\sum f_i \widehat{\lambda}_i^2 \\
& + C(A_2 + A_3)\Big(\sum f_i + F^\tau\Big) + CA_3 \\
\leq & -\frac{\beta A_1 N}{2\sqrt{n+1}}\Big(\sum f_k + F^\tau\Big) - \frac{A_3 c_0 b_0}{8}|\widehat{\lambda}| - A_3 \sum f_i|\widehat{\lambda}_i| \\
& + C(A_2 + A_3)\Big(\sum f_i + F^\tau\Big) + CA_3 \\
\leq & -\frac{\beta A_1 N}{2\sqrt{n+1}}\Big(\sum f_k + F^\tau\Big) - A_3\Big(1 + \sum f_i|\widehat{\lambda}_i|\Big) \\
& + C(A_2 + A_3)\Big(\sum f_i + F^\tau\Big)
\end{aligned} \tag{6.18}$$

by choosing $R \geq 8C/c_0 b_0 + 1$.

If $|\widehat{\lambda}| \leq R$, by (1.3) and (1.5), we have

$$c_1 I \leq \{F^{ij}\} \leq C_1, \quad c_1 \leq F^\tau \leq C_1$$

for some uniform positive constants c_1, C_1 which may depend on R. Therefore, we have

$$\mathcal{L}\Psi \leq C(-A_1 + A_2 + A_3)\Big(1 + \sum f_i + \sum f_i|\widehat{\lambda}_i| + F^\tau\Big) \tag{6.19}$$

where C depends on c_1 and C_1.

Case (**b**). By Lemma 2.1, we may fix a and δ sufficiently small such that $v \geq 0$ in M_T^δ and

$$\mathcal{L}v \leq -\frac{\varepsilon}{2}\Big(1 + \sum f_i + F^\tau\Big) \text{ in } M_T^\delta. \tag{6.20}$$

Thus, by Lemma 6.1, we have

$$
\begin{aligned}
\mathcal{L}\Psi \leq & -\frac{\varepsilon A_1}{2}\Big(1+\sum f_i + F^\tau\Big) + CA_2\sum f_i - \frac{A_3}{2}\sum_{i\neq r} f_i\widehat{\lambda}_i^2 \\
& + CA_3\Big(1+\sum f_i + \sum f_i|\widehat{\lambda}_i|\Big) \\
\leq & \Big(-\frac{\varepsilon A_1}{2} + CA_2 + CA_3\Big)\Big(1+\sum f_i + F^\tau\Big) - \frac{A_3}{4}\sum_{i\neq r} f_i\widehat{\lambda}_i^2 \\
\leq & \Big(-\frac{\varepsilon A_1}{2} + CA_2 + CA_3\Big)\Big(1+\sum f_i + F^\tau\Big) - A_3\sum f_i|\widehat{\lambda}_i|.
\end{aligned} \tag{6.21}
$$

Checking (6.18), (6.19) and (6.21), we can choose $A_1 \gg A_2 \gg A_3 \gg 1$ such that (6.3) holds and $\Psi \geq K(d+\rho^2)$ in $\overline{M_T^\delta}$. Therefore, Lemma 6.2 is proved. $\square$

The estimates for mixed tangential-normal second derivatives can be established immediately using Ψ as a barrier function by (6.4) and the maximum principle (see [3]).

The pure normal second derivatives can be derived as [9] using an idea of Trudinger [24]. The reader is referred to [3] for details.

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